

# THE DISCOVERY OF SQUEEZED STATES — IN 1927

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## Abstract

I first review a) the flowering of coherent states in the 1960's, yet b) the discovery of coherent states in 1926, and c) the flowering of squeezed states in the 1970's and 1980's. Then, with the background of the excitement over the then new quantum mechanics, I describe d) the discovery of squeezed states in 1927.

## 1 Coherent States

Coherent states are important in many fields of physics [1, 2]. This became widely recognized during the 1960's due to the work of Glauber [3], Klauder [4, 5], and Sudarshan [5, 6].

In modern parlance, they are standardly defined in three equivalent ways:

1) *Displacement-Operator Method*. For the harmonic oscillator, coherent states,  $|\alpha\rangle$ , are given by the unitary displacement operator acting on the ground state:

$$D(\alpha)|0\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \equiv |\alpha\rangle, \quad (1)$$

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] = \exp[-|\alpha|^2/2] \exp[\alpha a^\dagger] \exp[-\alpha^* a]. \quad (2)$$

The generalization of this method to arbitrary Lie groups has an involved history.<sup>2</sup> (See Refs. [7, 8] and later reviews [1, 2].) One simply applies the generalized displacement operator, which is the unitary exponentiation of the factor algebra, on to an extremal state.

2) *Ladder- (Annihilation-) Operator Method*. For the harmonic oscillator, the coherent states are also the eigenstates of the destruction operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \quad (3)$$

That these states are the same as the displacement-operator states follows from Eq. (1), since

$$0 = D(\alpha)a|0\rangle = (a - \alpha)D(\alpha)|0\rangle = (a - \alpha)|\alpha\rangle. \quad (4)$$

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<sup>2</sup>Unknown to many, Klauder developed this method in an early paper [7].

The generalization to arbitrary Lie groups is straight forward, and has also been studied [1, 2].

3) *Minimum-Uncertainty Method*. This method, as we will see in the next section, intuitively harks back to Schrödinger's discovery of the coherent states [9].

For the harmonic oscillator, the classical variables  $x$  and  $p$  vary as the sin and the cos of the classical  $\omega t$ . If one then takes these “natural” classical variables and transforms them into quantum operators, they define a commutation relation and an uncertainty relation:

$$[x, p] = i, \quad (\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4}. \quad (5)$$

The states that minimize this uncertainty relation are given by the solutions to the equation

$$\left( x + \frac{i}{2(\Delta p)^2} p \right) \psi_{mus} = \left( \langle x \rangle + \frac{i}{2(\Delta p)^2} \langle p \rangle \right) \psi_{mus}. \quad (6)$$

[Of the four parameters  $\langle x \rangle$ ,  $\langle x^2 \rangle$ ,  $\langle p \rangle$ , and  $\langle p^2 \rangle$ , only three are independent because the equality in the uncertainty relation is satisfied.] Rewrite Eq. (6) as

$$(x + iBp) \psi_{mus} = C \psi_{mus}, \quad B = \frac{\Delta x}{\Delta p}, \quad C = \langle x \rangle + iB \langle p \rangle. \quad (7)$$

Here  $B$  is real and  $C$  is complex. These states,  $\psi_{mus}(B, C)$ , are the minimum-uncertainty states.

$B$  can be adjusted to  $B_0$  so that the ground-eigenstate is a member of the set. Then these restricted states,  $\psi_{mus}(B = B_0, C) = \psi_{cs}(B_0, C)$ , are the minimum-uncertainty coherent states:

$$\psi_{cs} = \pi^{-1/4} \exp \left[ -\frac{(x - x_0)^2}{2} + ip_0 x \right]. \quad (8)$$

By the use of the Hermite polynomial generating function, and with the identifications

$$\alpha = \alpha_1 + i\alpha_2 = \frac{x_0 + ip_0}{\sqrt{2}}, \quad (9)$$

it can be shown that Eqs. (1) and (8) are equivalent. This method has been applied to general Hamiltonian potential and symmetry systems, yielding generalized coherent states [10].

The coherent wave packets are Gaussians, with widths that of the ground-state Gaussian. With time, the centroid of the packet follows the classical motion and does not change its shape.

## 2 The Discovery of Coherent States

The first half of 1926 was an amazingly productive period for Erwin Schrödinger. He submitted six important papers [9], beginning with his fundamental paper solving the hydrogen-atom.

There was, of course, tremendous controversy surrounding Schrödinger's work. In particular Lorentz wrote to Schrödinger on May 27 lamenting the fact that his wave functions were stationary, and did not display classical motion. On June 6 Schrödinger replied that he had found a system where classical motion was seen, and sent Lorentz a draft copy of the paper we are interested in, *Der stetige Übergang von der Mikro- zur Makromechanik* [9].

Using the generating function found in the classic book of Courant and Hilbert, Schrödinger realized that a Gaussian wave-function could be constructed from a particular superposition of the wave functions corresponding to the discrete eigenvalues of the harmonic oscillator. Further, these new states followed the classical motion. At this time the probability-*amplitude* nature of the wave function was not yet known, so the complex nature of the wave function bothered Schrödinger. He wondered if, perhaps, it was only the real part of the wave function that is physical.

But in any event, even though the uncertainty relation had yet to be discovered, from a view point that most closely resembles the modern minimum-uncertainty method, Schrödinger had discovered the coherent states.

During this period there was great rivalry between Schrödinger and the “young Turks,” Heisenberg and Pauli. Indeed, Pauli felt it necessary to mollify Schrödinger after having called his views, “local Zürich superstitions.”<sup>3</sup> Heisenberg was even stronger, saying Schrödinger’s physical picture makes scarcely any sense, “..in other words, I think it is crap.”<sup>4</sup> — Then again, Werner was hard to please. He later called the Dirac theory, “learned crap.”<sup>5</sup> (See Ref. [12].)

This period amusingly came to an end when Born corrected himself in proof. He realized that the probability was not proportional to the wave function, but to its square.

### 3 Squeezed States

Squeezed states were mathematically rediscovered and discussed by many authors [13], especially in the 1970’s and early 80’s. Each author tended to emphasize differing aspects of the system. Interest grew in squeezed states as it began to appear that experimentalists might be able to observe them [15]. This was first true in the field of gravitational-wave detection [16]. (Indeed, the term “squeezed states” was coined in the context of Weber-bar gravitational-wave detection [14].) But, especially with the quantum-optics experimental breakthroughs starting in the mid-1980’s [17], the field exploded, as evidenced also by the present conference series.

As coherent states, the more general squeezed states can be defined in three equivalent ways:

1) *Displacement-Operator Method*. To obtain squeezed states, one applies both the squeeze and displacement operators onto the ground state:

$$D(\alpha)S(z)|0\rangle = |\alpha, z\rangle, \quad z = re^{i\phi} = z_1 + iz_2, \quad S(z) = \exp\left[\frac{1}{2}za^\dagger a^\dagger - \frac{1}{2}z^*aa\right], \quad (10)$$

$$S(z) = \exp\left[\frac{1}{2}e^{i\phi}(\tanh r)a^\dagger a^\dagger\right] \left(\frac{1}{\cosh r}\right)^{(\frac{1}{2}+a^\dagger a)} \exp\left[-\frac{1}{2}e^{-i\phi}(\tanh r)aa\right]. \quad (11)$$

The ordering of  $DS$  vs.  $SD$  in Eq. (10) is equivalent, amounting to a change of parameters.

2) *Ladder- (Annihilation-) Operator Method*. Using a Holstein-Primakoff/Bogoliubov transformation,  $DSaS^{-1}D^{-1}$  is a linear combination of  $a$  and  $a^\dagger$ , that yields

$$\left[(\cosh r)a - (e^{i\phi} \sinh r)a^\dagger\right] |\alpha, z\rangle = \left[(\cosh r)\alpha - (e^{i\phi} \sinh r)\alpha^*\right] \cdot |\alpha, z\rangle \quad (12)$$

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<sup>3</sup>*Züricher Lokalaberglauben*

<sup>4</sup>*in a. W. ich finde es Mist.*

<sup>5</sup>*gelehrten Mist*

3) *Minimum-Uncertainty Method.* If one takes the convention  $z = r = \text{Real}$ , which is valid because  $\phi$  amounts to an initial time,  $t_0$ , one can return to Eq. (6) and realize that the  $\psi_{mus}$  are squeezed states. To include the phase explicitly, one moves a step further [18] than the Heisenberg uncertainty relation to the Schrödinger uncertainty relation [19]:

$$(\Delta x)^2(\Delta p)^2 \geq |\langle [x, p]/2 + \{x, p\}/2 \rangle|^2. \quad (13)$$

The states which satisfy this uncertainty relation, and which can be shown to be equivalent to the other formulations [20], are

$$\psi_{ss} = \frac{e^{-ix_0p_0/2}}{\pi^{1/4}[\mathcal{F}_1]^{1/2}} \exp \left[ -\frac{(x - x_0)^2}{2} \mathcal{F}_2 + ip_0x \right], \quad (14)$$

$$\mathcal{F}_1 = \cosh r + e^{i\phi} \sinh r, \quad \mathcal{F}_2 = \frac{\cosh r - e^{i\phi} \sinh r}{\cosh r + e^{i\phi} \sinh r}, \quad (15)$$

With time, the centroids of these wave packets also follow the classical motion, but they do not retain their shapes. The width of a particular Gaussian oscillates as

$$[\Delta x(t)]^2 = [(\cosh r)^2 + (\sinh r)^2 + 2(\cosh r)(\sinh r) \cos(2t - \phi)], \quad (16)$$

$$[\Delta p(t)]^2 = [(\cosh r)^2 + (\sinh r)^2 - 2(\cosh r)(\sinh r) \cos(2t - \phi)], \quad (17)$$

$$4[\Delta x(t)]^2[\Delta p(t)]^2 = 1 + \frac{1}{4} \left( s^2 - \frac{1}{s^2} \right)^2 \sin^2(2t - \phi), \quad s = e^r. \quad (18)$$

Generalizations have been given for other systems [21, 22]. Further, from Eq. (13), one can realize that generalized intelligent states [22] are, in fact, generalized squeezed states.

## 4 The Discovery of Squeezed States

In 1926 a Cornell University assistant professor, Earle Hesse Kennard (born 1885), was granted a sabbatical. Upon his return he would be promoted to full professor. In October Kennard arrived at Max Born's Institut für Theoretische Physik of the University of Göttingen, where Heisenberg and Jordan had also worked. There Kennard mastered the matrix mechanics of Heisenberg and the wave mechanics of Schrödinger.

At the same time, Heisenberg submitted his uncertainty relation paper and went to Copenhagen to work with Bohr. By the spring of 1927, Bohr was working on his famous Como talk on the uncertainty relation, which later became a special supplement to Nature. Kennard moved to Copenhagen on March 7. While there he completed the manuscript in which squeezed states were discovered [23], *Zur Quantenmechanik einfacher Bewegungstypen*. Kennard acknowledges the help of Bohr and Heisenberg in the paper. Also, in the grand European-professor tradition, Kennard left the paper with Bohr for Bohr to review and approve it, and then to submit it [24].

The paper itself begins with a review of matrix and wave mechanics. Then Kennard discusses the time-evolution of three systems: i) a particle in an electric field, ii) a particle in a magnetic field, and, so important to us, in Sec. 4C iii) a general Gaussian in a harmonic oscillator potential.

Kennard observes that the classical motion is followed and, in his own notation, gives the critical Eqs. (16)-(18) above. He also explains their meaning, i.e., that a) the sum of the uncertainties in  $x$  and  $p$  is a constant, b) the uncertainty product varies from the minimum as  $\sin^2 2t$  [ $\sin^2 t \cos^2 t$ ], c) the product has a minimum only twice each half period,<sup>6</sup> and d) when the uncertainties are equal (in natural units) then the (coherent) states of Schrödinger [9] are obtained. In other words, he had everything.

One can speculate why the papers of Schrödinger [9] and Kennard [23] were, in the main, ignored for so long. I think the simple answer is that they were too far ahead of their time. During the 20's, most would have felt it was inconceivable that this work meant more than a matter of principle. It would be decades before connection was made to experiment.

To be popular in physics you have to either be good or lucky. Sometimes it is better to be lucky. But if you are going to be good, perhaps you shouldn't be too good.

A more detailed version of this discussion will appear elsewhere [25].

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<sup>6</sup>Kennard wrote, *nur zweimal in jeder Schwingung*, so by *Schwingung* he meant “swing” instead of an entire oscillation.

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